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A note on Kolmogorov's third-order structure-function law, the local isotropy hypothesis and the pressure–velocity correlation

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We show that Kolmogorov's (1941*b*) inertial-range law for the third-order structure function can be derived from a dynamical equation including pressure terms and mean flow gradient terms. A new inertial-range law, relating the two-point pressure–velocity correlation to the single-point pressure–strain tensor, is also derived. This law shows that the two-point pressure–velocity correlation, just like the third-order structure function, grows linearly with the separation distance in the inertial range. The physical meaning of both this law and Kolmogorov's law is illustrated by a Fourier analysis. An inertial-range law is also derived for the third-order velocity–enstrophy structure function of two-dimensional turbulence. It is suggested that the second-order vorticity structure function of two-dimensional turbulence is constant and scales with $\epsilon_\omega^{2/3}$ in the enstrophy inertial range, ϵ_ω being the enstrophy dissipation. Owing to the constancy of this law, it does not imply a Fourier-space inertial-range law, and therefore it is not equivalent to the k^{-1} law for the enstrophy spectrum, suggested by Kraichnan (1967) and Batchelor (1969).

1. Introduction

Kolmogorov (1941*a,b*) developed the universal equilibrium theory for the small scales in turbulence by first making the hypothesis of 'local isotropy'. Local isotropy means that the statistical distribution of the velocity difference $\delta \boldsymbol{u} = \boldsymbol{u}' - \boldsymbol{u}$, of two points, is invariant under rotations and reflections, if the distance r between the points is small, that is if $r \ll L$, where L is the turbulence integral length scale. In his definition of local isotropy Kolmogorov also included steadiness in time of this distribution. Local isotropy implies that the nth-order statistical moment, or structure function,

$$B^{(n)}_{ij...k} = \langle \delta u_i \delta u_j \ldots \delta u_k \rangle, \tag{1.1}$$

is an isotropic tensor.

In his first paper (1941*a*) Kolmogorov introduced two similarity hypotheses for the locally isotropic turbulence field: first that the $\boldsymbol{B}^{(n)}$ of different orders are determined by the kinematic viscosity ν, the average dissipation rate ϵ and the distance r; secondly that if there is a range where $r \gg \eta = \nu^{3/4}/\epsilon^{1/4}$ and still $r \ll L$ – that is an inertial range – then the $\boldsymbol{B}^{(n)}$ are determined only by ϵ and r in this range.

In his second paper (1941*b*) Kolmogorov derived the inertial-range law

$$B^{(3)}_{lll}(r) = -\tfrac{4}{5}\epsilon r \tag{1.2}$$

for the third-order longitudinal structure function. Here the index l corresponds to the velocity component in the same direction as the separation vector $\boldsymbol{r}$ between the two points with velocities $\boldsymbol{u}$ and $\boldsymbol{u}'$. This is the only inertial-range scaling law that has been derived from the Navier–Stokes equations, and must therefore be considered to be a corner-stone of the theory. The fundamental importance of this law has been pointed out by Frisch (1991) and by Hunt & Vassilicos (1991), among others.

Kolmogorov (1941*b*) used the Kármán–Howarth (1938) equation for the two-point velocity correlation to derive (1.2). This equation presupposes global isotropy, or isotropy not only of the small scales of turbulence but also of the large scales. Therefore, it contains no pressure terms since these must be zero for globally isotropic turbulence. In the derivation of Landau & Lifshitz (1987) the pressure terms are set to zero with reference to isotropy. For the case when only local isotropy can be assumed, for example turbulence in a homogeneous shear flow, these terms cannot be *á priori* neglected. A more rigorous derivation of (1.2) should therefore start from an equation where they are retained and also include a demonstration of the assumptions that are needed to neglect them, if this is possible. An attempt along these lines has been made by Monin & Yaglom (1975). However, the pressure terms which appear in their derivation are by an erroneous argument set to zero with reference to local isotropy (see the Appendix), which makes it impossible to draw any well-founded conclusion about the behaviour of the two-point pressure correlation in the inertial range.

Here we will show that the two-point pressure–velocity correlation that is identically zero for a globally isotropic turbulence field and therefore set to zero in the Kármán–Howarth equation, is generally not negligible for the locally isotropic turbulence field in the inertial range, compared to the third-order structure tensor function. Actually, these quantities will be of the same order in the inertial range. This fact does not necessarily imply that the local isotropy hypothesis, as formulated by Kolmogorov, has to be given up; on the contrary it is perfectly consistent with the local isotropy hypothesis and enables us to derive a non-trivial inertial-range law for the two-point pressure–velocity correlation.

It would be desirable to convince ourselves of the general consistency of the local isotropy hypothesis, as we carry out the analysis. A natural way to do this is to introduce the Reynolds decomposition

$$\boldsymbol{u} = \boldsymbol{U} + \tilde{\boldsymbol{u}}, \tag{1.3}$$

where $\boldsymbol{U}$ is the mean velocity and $\tilde{\boldsymbol{u}}$ is the fluctuating part of the velocity. According to the assumption of the small-scale independence of the large scales, the $\boldsymbol{B}^{(n)}$ must be dynamically independent of the mean shear for small separations. It is also clear that the structure functions $\tilde{\boldsymbol{B}}^{(n)}$, of the fluctuating part of the velocity, with the same assumption of the small-scale independence, can be equated to the $\boldsymbol{B}^{(n)}$ with a very high degree of accuracy. For the homogeneous case we have

$$\delta U_i = r_j \frac{\partial U_i}{\partial x_j}, \tag{1.4}$$

where $\partial U_i/\partial x_j$ is the mean flow gradient tensor. The local isotropy hypothesis presupposes that the variation of the mean flow within a small domain will not influence turbulence structures confined in the same domain. Thus, the condition for putting $\tilde{\boldsymbol{B}}^{(n)} = \boldsymbol{B}^{(n)}$,

$$\left| r_j \frac{\partial U_i}{\partial x_j} \right| \ll \langle \delta u_i \delta u_i \rangle^{1/2}, \tag{1.5}$$

must be fulfilled when local isotropy is the case. As we shall see, this condition is well satisfied if r is in the inertial range or smaller.

The problem that $\delta \boldsymbol{u}$ generally contains a non-random component has been recognized by Monin & Yaglom (1975, p. 102). To overcome the problem they suggest a transformation whose effect is a replacement of $\delta \boldsymbol{u}$ by $\delta \tilde{\boldsymbol{u}}$. We follow this suggestion. To convince ourselves that the small scales are dynamically independent of the mean shear, we formulate the dynamical equation for $\tilde{\boldsymbol{B}}^{(n)}$, rather than for $\boldsymbol{B}^{(n)}$, and investigate the possible influence from the mean flow gradient terms. This approach has also a second advantage. The turbulent energy is produced in the large scales through interaction with the mean shear and is dissipated in the very smallest scales. The Reynolds decomposition makes this dynamical picture clear. By using an energy equation with explicit production terms including the mean flow gradient, the relation between production and dissipation is revealed. Henceforth, we omit the tilde from quantities describing the fluctuating part of the hydrodynamic field.

2. Derivation of the inertial-range laws

We will now derive (1.2) for an incompressible homogeneous shear flow, in which case the dynamical equation for the second-order two-point correlation tensor reads (Hinze 1975)

$$\begin{aligned}\frac{\partial}{\partial t}\langle u_i u_j' \rangle = & -\frac{\partial}{\partial r_s}(\langle u_i u_s' u_j' \rangle - \langle u_i u_s u_j' \rangle) \\ & -\frac{\partial U_m}{\partial x_s}\left(\delta_{mi}\langle u_s u_j' \rangle + \delta_{mj}\langle u_s' u_i \rangle + r_s \frac{\partial}{\partial r_m}\langle u_i u_j' \rangle\right) \\ & -\frac{1}{\rho}\left(\frac{\partial}{\partial r_j}\langle u_i p' \rangle - \frac{\partial}{\partial r_i}\langle u_j' p \rangle\right) + 2\nu \frac{\partial^2}{\partial r_s \partial r_s}\langle u_i u_j' \rangle,\end{aligned} \tag{2.1}$$

where repeated indices are contracted. The corresponding single-point equation reads

$$\frac{\partial}{\partial t}\langle u_i u_j \rangle = \Psi_{ij} + \Pi_{ij} - \epsilon_{ij}, \tag{2.2}$$

where the tensors of the right-hand side are defined respectively as the single-point limit of the mean flow gradient terms, the pressure terms and the viscous term of (2.1). For the viscous term the sign has also been reversed compared to (2.1). $\boldsymbol{\Psi}$ is the production tensor which is commonly denoted by '$\boldsymbol{P}$'. Here we adopt the convention of using greek letters for single-point correlations and latin letters for two-point correlations. $\boldsymbol{\Pi}$ is the pressure–strain tensor and $\boldsymbol{\epsilon}$ is the dissipation tensor. Half the trace of $\boldsymbol{\epsilon}$ is equal to the average dissipation rate ϵ.

By adding to (2.1) the corresponding equation where i and j are switched and by using (2.2), we find without any approximations

$$\begin{aligned}2\Pi_{ij} - 2\epsilon_{ij} - \frac{\partial}{\partial t}B_{ij}^{(2)} = & \frac{\partial}{\partial r_s}B_{sij}^{(3)} \\ & +\frac{\partial U_i}{\partial x_s}B_{sj}^{(2)} + \frac{\partial U_j}{\partial x_s}B_{si}^{(2)} + \frac{\partial U_m}{\partial x_s}r_s\frac{\partial}{\partial r_m}B_{ij}^{(2)} \\ & -\frac{\partial}{\partial r_j}P_i - \frac{\partial}{\partial r_i}P_j - 2\nu\frac{\partial^2}{\partial r_s \partial r_s}B_{ij}^{(2)},\end{aligned} \tag{2.3}$$

where we have introduced the notation

$$P_i = \frac{1}{\rho}\left(\langle u_i p'\rangle - \langle u_i' p\rangle\right) \tag{2.4}$$

for the two-point pressure–velocity correlation. To go from (2.1) and (2.2) to (2.3) homogeneity and incompressibility have been used.† It can be of some interest to note that the pressure terms on the right-hand side of (2.3), unlike all the other two-point correlation terms, cannot be reduced to expressions of structure-function form, such as

$$\frac{1}{\rho}\frac{\partial}{\partial r_i}\langle \delta p \delta u_j\rangle . \tag{2.5}$$

Like all other terms in (2.3) the pressure terms are by homogeneity invariant under a change of sign of $\boldsymbol{r}$, while (2.5) changes sign if the sign of $\boldsymbol{r}$ is changed. No such term as (2.5) can therefore appear in equation (2.3).

Kolmogorov (1941*b*) obtained (1.2) by rewriting the Kármán–Howarth equation, which is the isotropic form of (2.1), into a form similar to (2.3) and integrating it from zero separation to the inertial range. For the globally isotropic turbulence field the pressure terms and the mean flow gradient terms of (2.1) are zero. By comparing the first two terms of the left-hand side of (2.3) we can see that for the locally isotropic turbulence field the pressure terms cannot be *á priori* neglected. The components of the pressure–strain tensor can generally not be assumed to be small compared to the components of the dissipation tensor. It is reasonable to assume that for many realistic flows $|\boldsymbol{\Pi}|$ and ϵ are of the same order of magnitude. In direct numerical simulations of a homogeneous shear flow ($R_\lambda \approx 100$), Rogers, Moin & Reynolds (1986) found that the magnitude of the diagonal components of $\boldsymbol{\Pi}$ were of the same order as the corresponding components of ϵ. From measurements in a nearly homogeneous shear flow with $R_\lambda \approx 150$, Harris, Graham & Corrsin (1977) estimated $\boldsymbol{\Pi}$, and found components of the same order as ϵ. It is also a reasonable assumption that $|\boldsymbol{\Pi}| \sim \epsilon$ in the limit of infinite Reynolds number. Assuming perfect stationarity and isotropic dissipation and using equation (7), we obtain

$$\Pi_{22} = \Pi_{33} = \tfrac{2}{3}\epsilon, \qquad \Pi_{11} = -\tfrac{4}{3}\epsilon, \tag{2.6}$$

where the indices refer to a coordinate system in which $\partial U_1/\partial x_2$ is the only non-zero component of the mean flow gradient tensor. Rogers *et al.* (1986) obtained results not very far from (2.6) in the middle of the simulations when the flow field had been developed for some time, while the Reynolds number still was not too low. The experimental results of Harris *et al.* (1977) differ from (2.6) by about 30%.

In the single-point limit the two-point pressure terms on the right-hand side of (2.3) will be identical with the pressure–strain tensor of the left-hand side. So for

† A demonstration of how the triple correlation terms are rewritten can be found in Frisch (1995). This book was published during the final revision of this paper and contains a derivation of (1.2), (F95). There are some similarities between our derivation and F95. The triple correlation terms are rewritten in the same way and the close connection between Kolmogorov's law and a Fourier description of the nonlinear energy flow, is pointed out in both derivations. There are also several differences. F95 does not make a Reynolds decomposition of the flow field, as we do. The pressure terms are not treated by F95, since only the trace of (2.1) is studied. The single-point equation (2.2) is not used by F95 to rewrite (2.1) into structure function form. Only the triple correlation terms are rewritten. To derive (1.2), F95 assumes stationarity in the global energy equation which means that the time derivative in (2.2) can be omitted. We apply the assumption of stationarity to the equation (2.3). According to Kolmogorov's definition of local isotropy the time derivative in (2.3) is equal to zero.

small separations they cannot be neglected. The mean flow gradient terms of (2.3) can on the other hand be assumed to be small for small separations. If the energy production is assumed to be approximately equal to the dissipation we can estimate the order of magnitude of the mean flow gradient as

$$\left|\frac{\partial U_m}{\partial x_s}\right| \sim \frac{\epsilon}{\langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle}. \tag{2.7}$$

For small separations, in the inertial range and smaller, we have

$$\frac{|\,\boldsymbol{B}^{(2)}\,|}{\langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle} \ll 1. \tag{2.8}$$

The mean flow gradient terms of (2.3) are therefore negligible compared to ϵ, which is the leading order of (2.3) in the inertial range. However, the mean flow gradient terms could not have been neglected already in the equation (2.1) with the same result, since they contribute to the single-point equation (2.2) with the dynamically very important production tensor $\boldsymbol{\Psi}$. The single-point terms in (2.3) were obtained by making use of equation (2.2) in which the production tensor is an important term.

We shall now integrate (2.3) over the volume of a sphere with radius r, where r is in the inertial range. By definition, the viscous term is small in the inertial range. By neglecting this term, as well as the time derivative and the mean flow gradient terms, we find by virtue of the divergence theorem

$$\frac{8\pi r}{3}\left(\Pi_{ij} - \epsilon_{ij}\right) = \int \left(n_s B^{(3)}_{sij} - n_j P_i - n_i P_j\right) \mathrm{d}\Omega, \tag{2.9}$$

where $\boldsymbol{n} = \boldsymbol{r}/r$, and $\mathrm{d}\Omega$ is the element of solid angle. Each of the neglected terms can, when integrated and divided by r^2, also be written as integrals of the same form as in (2.9). The neglected viscous term can be written as

$$-2\nu \int n_s \frac{\partial}{\partial r_s} B^{(2)}_{ij}\, \mathrm{d}\Omega. \tag{2.10}$$

By virtue of the incompressibility condition, the time derivative can be written as

$$-\frac{\partial}{\partial t}\int n_j r_s B^{(2)}_{si}\, \mathrm{d}\Omega, \tag{2.11}$$

and the mean flow gradient terms as

$$\int \left[\left(\frac{\partial U_i}{\partial x_s} n_j + \frac{\partial U_j}{\partial x_s} n_i\right) r_m B^{(2)}_{sm} + \frac{\partial U_m}{\partial x_s} n_m r_s B^{(2)}_{ij}\right] \mathrm{d}\Omega. \tag{2.12}$$

The terms in (2.10)–(2.12) can thus be compared with the terms in (2.9) only with reference to inertial-range quantities. If the concept of an inertial range is to have any relevance the terms in (2.10)–(2.12) must be negligible. If we use the Kolmogorov (1941*a*) similarity law $|\,\boldsymbol{B}^{(2)}\,| \sim \epsilon^{2/3} r^{2/3}$ together with an assumption of quasi-stationarity we indeed find them small.

If $|\,\boldsymbol{\Pi}\,| \sim \epsilon$, so that the two left-hand-side terms of (2.9) are of the same order of magnitude, then it is reasonable to assume that the two different types of terms on the right-hand side of (2.9) also are of the same order of magnitude, so that

$$|\,\boldsymbol{B}^{(3)}\,| \sim \epsilon r, \tag{2.13}$$

$$|\,\boldsymbol{P}\,| \sim \epsilon r, \tag{2.14}$$

in the inertial range. These are the strongest conclusions we can draw from (2.9) without introducing the local isotropy hypothesis.

By taking the trace of (2.9), the pressure terms of both the left-hand side and the right-hand side disappear, since they are traceless due to the condition of incompressibility. Thus we find

$$-\frac{16\pi r}{3}\epsilon = \int n_s B^{(3)}_{sii}\,\mathrm{d}\Omega\,. \tag{2.15}$$

If we now assume that the vector $B^{(3)}_{sii}$ is isotropic, then the integrand of (2.15) is independent of angle and we immediately find that

$$n_s B^{(3)}_{sii} = -\tfrac{4}{3}\epsilon r \tag{2.16}$$

in the inertial range. If we further assume that the tensor $\boldsymbol{B}^{(3)}$ is isotropic, which of course is a much stronger assumption, then any component of $\boldsymbol{B}^{(3)}$ can be uniquely related to (2.16), since $\boldsymbol{B}^{(3)}$ in this case has only one independent component. Each component of $\boldsymbol{B}^{(3)}$ can by isotropy and index-symmetry be determined by the two components $B^{(3)}_{lll}$ and $B^{(3)}_{ltt}$, where t indicates a direction perpendicular to $\boldsymbol{r}$. These two components are related through incompressibility by

$$B^{(3)}_{ltt} = \frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}r}\left(rB^{(3)}_{lll}\right) \tag{2.17}$$

(Landau & Lifshitz 1987). Clearly we have

$$n_s B^{(3)}_{sii} = B^{(3)}_{lll} + 2B^{(3)}_{ltt} \tag{2.18}$$

(with contraction over the indices s and i, but not over l and t which correspond to specific directions). From (2.16)–(2.18) Kolmogorov's law (1.2) follows, and also the relation

$$B^{(3)}_{ltt}(r) = -\tfrac{4}{15}\epsilon r \tag{2.19}$$

In this derivation of (1.2) (or (2.19)) the dynamical equation (2.1) is not forced into its isotropic form from the start, which has the double advantage of making the derivation more rigorous and considerably simpler. Putting (2.1) into its isotropic form involves some rather tedious manipulations, using the conditions of isotropy and incompressibility (see Kármán & Howarth 1938 or Landau & Lifshitz 1987). Here we proceed in a few steps to (2.15) without introducing isotropy. This procedure emphasizes the convective nonlinearity of the Navier–Stokes equations, rather than any specific isotropic relations, as the main condition for deriving a scaling law for the third-order structure function.

From (2.9) it can be seen that the local isotropy hypothesis also implies another inertial range law. The dissipation tensor $\boldsymbol{\epsilon}$ of the left-hand side of (2.9) is isotropic for the locally isotropic turbulence field, as well as $\boldsymbol{B}^{(3)}$ when r is in the inertial range. The pressure terms of both the left-hand side and the right-hand side of (2.9) are traceless and thus they contain no isotropic part. If local isotropy holds, the pressure terms must therefore balance each other. This yields

$$\frac{8\pi r}{3}\Pi_{ij} = -\int \left(n_j P_i + n_i P_j\right)\,\mathrm{d}\Omega \tag{2.20}$$

in the inertial range.

From one point of view the relation (2.20) might seem trivial. In the single-point limit the pressure terms of the two sides of (2.3) become identical. Therefore, (2.20) must also hold for very small separations, of the order of the Kolmogorov scale

η, since (2.20) is basically equivalent to the first-order Taylor expansion around zero separation. (An explicit Taylor expansion for scales of order η and smaller is somewhat more general than (2.20), but from our derivation it is clear that the special form of (2.20), arising from our choice of a sphere as the integration volume, is not crucial here, since we could have integrated over any other simple closed volume.) If the inertial-range separations also were considered as 'small', then it could be asserted that (2.20) can be anticipated without any use of the dynamical equation, thus being rather trivial. But this is not true. The inertial-range separations are small only compared to the integral length scale L, while they are large compared to η. The two-point velocity correlations, such as for example $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$, can only be estimated accurately by a low-order Taylor expansion out to separations of the order of η, while the inertial range is far beyond the range where such an expansion is valid. Here we have found that $\boldsymbol{P}$, unlike the velocity correlations, can be estimated accurately in the inertial range by the lowest-order expansion, if local isotropy holds. The inertial range grows wider with increasing Reynolds number, and consequently this is also true for the range where the expansion is supposed to hold.

From another point of view the relation (2.20) might seem somewhat paradoxical. A two-point quantity $\boldsymbol{P}$ which is identically zero for the globally isotropic turbulence field, must grow linearly with the separation distance in the inertial range; it must also be of the same order of magnitude as $\boldsymbol{B}^{(3)}$, if the turbulence field is to be locally isotropic. The idea of local isotropy is that the small structures of turbulence look identically the same in all directions. Adhering to this idea we cannot interpret (2.20) as saying anything about turbulence structures in the inertial-range. This is true even though it is an inertial range law, in the meaning that it determines the behaviour of a two-point correlation for separation distances that lie in this range. That (2.20) is not in direct conflict with local isotropy, can be understood from the fact that it cannot be formulated as a law for the pressure–velocity structure function $\langle \delta p \delta \boldsymbol{u} \rangle$. In the definition of local isotropy it is not unreasonable to include that $\langle \delta p \delta \boldsymbol{u} \rangle$ must be isotropic for $r \ll L$. The isotropic form of such a structure function is zero (Monin & Yaglom 1975, p. 103). But (2.20) is by no means incompatible with such a condition.

In the next section, where we develop our analysis in Fourier space, we shall give (2.20) an interpretation that is neither trivial nor paradoxical, but in full agreement with the idea behind local isotropy.

3. Fourier analysis of the inertial-range laws

First we define the Fourier transform of $\boldsymbol{P}$ as

$$\widehat{\boldsymbol{P}}(\boldsymbol{k}) = \frac{1}{(2\pi)^3} \int \boldsymbol{P}(\boldsymbol{r}) \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r})\, \mathrm{d}^3 r, \tag{3.1}$$

with the inverse transform

$$\boldsymbol{P}(\boldsymbol{r}) = \int \widehat{\boldsymbol{P}}(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r})\, \mathrm{d}^3 k. \tag{3.2}$$

The Fourier transforms of other two-point correlations are defined in the same way.

The pressure–strain tensor $\boldsymbol{\Pi}$, being defined as the single-point limit of the two-point pressure–velocity correlation terms of the dynamical equation (2.1), can be written as

$$\Pi_{ij} = -\frac{\mathrm{i}}{2} \int \left(k_j \widehat{P}_i + k_i \widehat{P}_j \right) \mathrm{d}^3 k. \tag{3.3}$$

The pressure–strain tensor undoubtedly has the role of transferring parts of the energy content of some directional components to other components. Thus, it reflects the tendency of the turbulence to isotropize. If the process of isotropization mainly takes place in the large scales, then the integral (3.3) will be dominated by contributions from wave vectors $\boldsymbol{k}$ for which $k = |\boldsymbol{k}| < k_0$, where k_0 is a wavenumber at the lower end of the Fourier-space inertial range, which is defined as the region where $\eta \ll 1/k \ll L$. This type of reasoning rests on the commonly accepted idea of a correspondence between the size of turbulence structures and wavenumber. The assumption of the dominance of large scales to the integral (3.3) is in full agreement with the local isotropy hypothesis. By operating on (2.3) with $\partial/\partial r_j$ we find

$$\nabla^2 P_i = \frac{\partial^2}{\partial r_s \partial r_j} B^{(3)}_{sji} + 2\frac{\partial U_j}{\partial x_s}\frac{\partial}{\partial r_j} B^{(2)}_{si} . \tag{3.4}$$

In order to express the right-hand side of (3.4) in terms of correlations that go to zero when r goes to infinity, we can change $B^{(2)}_{si}$ in (26) to $-R_{si} - R_{is}$, where $R_{is} = \langle u_i u'_s \rangle$. These expressions differ only by a constant which is of no relevance in (3.4). In Fourier space the solution to (3.4) can be written

$$\widehat{P}_i = \frac{k_s k_j}{k^2} \widehat{B}^{(3)}_{sji} + 2\frac{\partial U_j}{\partial x_s}\frac{\mathrm{i}k_j}{k^2}\left(\widehat{R}_{is} + \widehat{R}_{si}\right) , \tag{3.5}$$

where $\widehat{\boldsymbol{R}}$ can be identified as the energy spectrum tensor. Substituting the expression (3.5) into (3.3) we find that the part of the integral related to mean flow gradient is weighted on the energy-bearing small wavenumbers. For this part the contribution from wavenumbers greater than k_0 must be very small. For the other part, related to $\widehat{\boldsymbol{B}}^{(3)}$, a sufficient (but not necessary) condition for the contribution to the total integral (3.3) from wavenumbers greater than k_0 to be zero, is that $\widehat{\boldsymbol{B}}^{(3)}$ is isotropic for these wavenumbers. If this is the case, then the vector $k_s k_j \widehat{B}^{(3)}_{sji}$ is zero, due to incompressibility, and consequently the first term of the right-hand side of (3.5) will also be zero. The assumption that the isotropization takes place in large scales and that the integral (3.3) therefore is dominated by wavenumbers less than k_0, is clearly closely connected to the local isotropy hypothesis.

We shall now establish the connection between the inertial-range law (2.9) and a Fourier description of the energy transfer from large to small scales. First, we take the Fourier transform of equation (2.3) and find, with the approximations that led to (2.9), that in the inertial range

$$\mathrm{i}\left(k_s \widehat{B}^{(3)}_{sij} - k_j \widehat{P}_i - k_i \widehat{P}_j\right) \approx 0 . \tag{3.6}$$

This relation is basically equivalent to the perhaps more familiar statement that 'the energy transfer function is zero in the inertial range'.

Now we express $\boldsymbol{P}$ and $\boldsymbol{B}^{(3)}$ as Fourier integrals, and substitute these expressions into (2.9). This yields

$$\frac{8\pi r}{3}\left(\Pi_{ij} - \epsilon_{ij}\right) = \int\int \left(n_s \widehat{B}^{(3)}_{sij} - n_j \widehat{P}_i - n_i \widehat{P}_j\right) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\, \mathrm{d}^3k\, \mathrm{d}\Omega . \tag{3.7}$$

Integrating in the angular variable Ω and dividing by $8\pi/3$ we find

$$r\left(\Pi_{ij} - \epsilon_{ij}\right) = \frac{3\mathrm{i}}{2}\int \frac{j_1(kr)}{k}\left(k_s \widehat{B}^{(3)}_{sij} - k_j \widehat{P}_i - k_i \widehat{P}_j\right) \mathrm{d}^3k , \tag{3.8}$$

where j_1 is the first-order spherical Bessel function. When k is in the inertial range (of

Fourier space) then the integrand of (3.8) is approximately zero, due to relation (3.6). When k is larger than the inertial-range wavenumbers then $kr \gg 1$ and j_1 oscillates rapidly. The contribution from this region to the integral (3.8) must therefore be negligible. Hence, the integral (3.8) is dominated by the region where $k < k_0$. If r lies well inside the inertial range (of real space) then $kr \ll 1$ for these wavenumbers, and the Bessel function can be expanded, $j_1(kr) = kr/3 + O((kr)^3)$. Thus, we find to the lowest order

$$\Pi_{ij} - \epsilon_{ij} = \frac{\mathrm{i}}{2} \int_{k<k_0} \left(k_s \widehat{B}^{(3)}_{sij} - k_j \widehat{P}_i - k_i \widehat{P}_j \right) \mathrm{d}^3 k . \tag{3.9}$$

This is basically the same law as (2.9), but formulated in Fourier space. While (2.9) is valid in the inertial range of real space, (3.9) falls totally outside the inertial range of Fourier space. Therefore (2.9) does not necessarily say anything about turbulence structures of sizes in the inertial range. The same can of course be said about the relation (2.20). By comparing (3.9) with (3.3) we see that the pressure terms of each side of (3.9) balance each other if (and only if) the integral of (3.3) is dominated by the region where $k < k_0$. In this case (2.20) must hold, and thus we can interpret this law as a consequence of the hypothesis that the isotropization takes place in the large scales of turbulence.

If the pressure terms of (3.9) balance each other we must also have

$$\tfrac{1}{2}\epsilon_{ij} = -\frac{\mathrm{i}}{4} \int_{k<k_0} k_s \widehat{B}^{(3)}_{sij} \, \mathrm{d}^3 k . \tag{3.10}$$

This relation states that the flow of energy from small wavenumbers into the inertial range of Fourier space is equal to the amount of dissipated energy, and that if the tensor ϵ is isotropic, i.e. if an equal amount of energy is dissipated in each directional component, then the flow of energy into the inertial range must also be equally distributed over the components. From our derivation it is clear that (3.10) is basically the same law as Kolmogorov's law (1.2), but formulated in Fourier space.

4. Two-dimensional turbulence

From a geometrical point of view our derivation of Kolmogorov's law (1.2) could have been repeated for the two-dimensional case, resulting in a similar relation, differing from (1.2) only by a numerical factor. Instead of integrating over a sphere as in the three-dimensional case, we should have integrated over a circle in the two-dimensional case. A relation corresponding to (3.10) could also have been derived, but in the two-dimensional case a cylindrical first-order Bessel function would have appeared instead of the spherical Bessel function in (3.10). But (3.10) cannot hold if there is a backward cascade of energy from large to small wavenumbers, as is the case in two-dimensional turbulence (Kraichnan 1967). Therefore, no such relation as (1.2) can be true for this case. Some of the assumptions that led to (1.2) must therefore be incorrect for two-dimensional turbulence, and the obvious candidate is the assumption of stationarity of $\boldsymbol{B}^{(2)}$, that led to the neglect of (2.12). If there is a backward cascade of energy, this assumption cannot be expected to hold.

In two-dimensional turbulence there is a forward cascade of enstrophy (half the square of the vorticity) (Kraichnan 1967). Searching for the two-dimensional counterpart of (1.2) we therefore study the vorticity equation, which in the two-dimensional incompressible plane case can be written

$$\frac{\partial \omega}{\partial t} + u_s \frac{\partial \omega}{\partial x_s} = \nu \nabla^2 \omega, \tag{4.1}$$

where ω is the (only non-zero) vorticity component, which points out from the plane. For the homogeneous case we can from (4.1) derive an equation for the two-point vorticity correlation function:

$$\frac{\partial}{\partial t}\langle \omega\omega' \rangle - \frac{\partial}{\partial r_s}(\langle u_s \omega\omega' \rangle - \langle u_s' \omega' \omega \rangle) = 2\nu \frac{\partial^2}{\partial r_s \partial r_s}\langle \omega\omega' \rangle, \tag{4.2}$$

corresponding to equation (2.1) of §2. By using the condition of incompressibility and by rearranging the terms in (4.2), we can derive the relation

$$-4\epsilon_\omega - \frac{\partial}{\partial t}\langle \delta\omega\delta\omega \rangle = \frac{\partial}{\partial r_s}\langle \delta u_s \delta\omega\delta\omega \rangle - 2\nu \frac{\partial^2}{\partial r_s \partial r_s}\langle \delta\omega\delta\omega \rangle, \tag{4.3}$$

where

$$\epsilon_\omega = \nu \left\langle \frac{\partial \omega}{\partial x_i}\frac{\partial \omega}{\partial x_i} \right\rangle \tag{4.4}$$

is the enstrophy dissipation, and where we have assumed that there is no mean vorticity gradient. Assuming that there is a range of separations where the time-derivative and the viscous term of (4.3) can be neglected, we find by integrating over the area of a circle lying in this range:

$$\int_0^{2\pi} n_s \langle \delta u_s \delta\omega\delta\omega \rangle \, \mathrm{d}\phi = -4\pi\epsilon_\omega r. \tag{4.5}$$

Isotropy implies that the integrand is independent of angle and

$$n_s \langle \delta u_s \delta\omega\delta\omega \rangle = -2\epsilon_\omega r, \tag{4.6}$$

which is the two-dimensional counterpart of Kolmogorov's law (1.2). From (4.6) it is also possible to derive an equation exactly corresponding to (3.10), with the corresponding interpretation that the flow of enstrophy into the inertial range of Fourier space is equal to ϵ_ω.

For two-dimensional turbulence the natural analogy to Kolmogorov's (1941*a*) similarity hypothesis for the velocity structure functions is a corresponding hypothesis for the even-order vorticity structure functions, with ν and ϵ_ω as scaling parameters. The odd-order vorticity structure functions must be zero for plane two-dimensional turbulence, owing to reflectional symmetry in the plane. In the inertial range, that is where $\eta_\omega = \nu^{1/2}/\epsilon_\omega^{1/6} \ll r \ll L_\omega$, where L_ω is an enstrophy integral length scale, the hypothesis gives

$$\langle \delta\omega\delta\omega \rangle = C_\omega \epsilon_\omega^{2/3}, \tag{4.7}$$

where C_ω is a constant. The similarity hypothesis for the second-order velocity structure function of three-dimensional turbulence can be translated to Fourier space, giving us the famous $k^{-5/3}$ law for the energy spectrum. No such prediction of the enstrophy spectrum can be based on the hypothesis (4.7), since the Fourier transform of a constant is a delta-function (see for example Lighthill 1959):

$$\frac{1}{(2\pi)^2}\int C_\omega \epsilon_\omega^{2/3} \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \, \mathrm{d}^2 k = \delta(\boldsymbol{k}) C_\omega \epsilon_\omega^{2/3}. \tag{4.8}$$

Therefore the hypothesis (4.7) is not equivalent to the Fourier-space inertial-range law for the enstrophy spectrum,

$$\Phi(k) = C\epsilon_{\omega}^{2/3}k^{-1}, \tag{4.9}$$

suggested by Kraichnan (1967) and Batchelor (1969).

The hypothesis (4.7) may easily be refined in the same way as Kolmogorov's 1941 hypothesis was refined by Obukhov (1962) and Kolmogorov (1962), while it is impossible to reformulate (4.9) in direct analogy with the Obukhov–Kolmogorov 1962 theory. In the refined formulation of (4.7) ϵ_{ω} must be replaced by ϵ_{ω_r} – the average of the enstrophy dissipation over a disk of radius r.

The numerical evidence in favour of (4.9) does not seem to be entirely conclusive. In a high-resolution direct numerical simulation of freely decaying two-dimensional turbulence Brachet *et al.* (1988) found energy spectra decreasing with exponents varying between -3 and -4, corresponding to exponents between -1 and -2 for the enstrophy spectrum. Whether (4.7) or (4.9) holds, or neither of them, must be decided by further experiments, or by some physical argument yet unknown.

5. Concluding remarks

Our analysis of the dynamical equation for the velocity two-point correlations of a homogeneous shear flow has shown that the concept of an inertial range where the third-order structure function generally scales according to (2.13), can be developed from an equation with all the dynamically relevant terms retained. Furthermore, it has been shown that the two-point pressure–velocity correlation scales according to (2.14) in this range, if the pressure–strain is of the same order as the dissipation. The introduction of the local isotropy hypothesis made it possible to formulate these relations as exact laws, as in (1.2) or (2.19), and (2.20). By a Fourier analysis it was shown that Kolmogorov's law (1.2) is closely connected to the notion of an energy flow from large scales into the inertial range, and that the corresponding law (2.20) for the pressure–velocity correlation can be interpreted as a consequence of a hypothesis that isotropization takes place only in the large scales of turbulence.

Our analysis has neither proved nor disproved the local isotropy hypothesis. The most we can say is that the hypothesis seems to be fully consistent with the Navier–Stokes equations. It is difficult to see that the dynamical equation (2.1) could be used more thoroughly than it has been used here, to decide on the matter of local isotropy. Therefore it is our belief that the decision has to be left to experiment.

A direct experimental verification of the inertial-range law (2.20) for the pressure–velocity correlation, is practically impossible with the experimental techniques used today. Up till now it has been considered impossible to measure the pressure–strain by other means than measuring the other quantities in the equation (2.2) and estimating it as the remainder of the energy balance. Recent theoretical development (Lindborg 1995) of the kinematical theory of homogeneous axisymmetric turbulence, has made it clear that it is possible to measure the pressure–strain and in principle also the two-point pressure–velocity correlations, by measuring velocity correlations and then solving a Poisson equation. Substantial progress along these lines has also been made in experiments by Sjögren & Johansson (private communication) at KTH (Royal Institute of Technology, Sweden). However, the limitations set by the size of the wind tunnel and hot wires, make it impossible to reach the high Reynolds numbers for which (2.20) can be supposed to hold. This relation can only be indirectly verified by an experimental verification of the isotropic relations (1.2) and (2.19). If these hold, then the third-order structure tensor function is isotropic in the inertial range, and the pressure terms in (2.9) must balance each other, as expressed by (2.20). As far as we know there are no reported experimental verifications of (2.19), while there are several verifications of (1.2). The results from measurements of the third-order longitudinal structure function by Van Atta & Chen (1970), at different heights in an atmospheric boundary layer over the ocean, are in quite good agreement with (1.2); the result of Van Atta & Park (1980) from a marine boundary layer is in very good agreement with (1.2), and the results of Antonia, Zhu & Hosokawa (1995) from an atmospheric boundary layer and a circular jet are in reasonable agreement with (1.2).

Recently an impressive experimental investigation of the local isotropy hypothesis has been performed by Saddoughi & Veeravalli (1994) in a high Reynolds number ($R_\lambda = 500$–1450) boundary layer. By using hot-wire x-probes they measured streamwise energy, and dissipation spectra of different velocity components and found an increasing agreement (in these measures) with isotropic relations with increasing wavenumber in the inertial range, while the agreement appeared to be almost perfect in the dissipation range. Their conclusion is entirely in favour of the local isotropy hypothesis. There is, however, a question to be asked regarding their measurements. The total dissipation was first measured by integration of the measured dissipation spectra under the assumption of local isotropy. No spanwise or wall-normal spectra were measured. The error in this value due to statistical scatter can be estimated from their plots to be less than a few per cent. The dissipation was then calculated by using the relation (1.2), to which the measured structure function fitted rather well for more than one decade of separations. The value measured in this way was found to be more than 20% lower than the one measured in the first way. Unfortunately there is no discussion of the possible reason for this rather large deviation in their paper. One explanation could be that the dissipation in fact suffered from an anisotropy affecting the wall-normal spectra more significantly than the measured streamwise spectra, and that the value measured in the first way therefore was wrong. Another explanation could be that the third-order structure tensor function was not isotropic in the inertial range and that the value calculated by using (1.2) which presupposes isotropy therefore was wrong. It might be that the relation (2.20) was not fulfilled and that influences of pressure forces in the inertial range therefore affected the scaling constant of the longitudinal third-order structure function. It is a pity that the other (longitudinal-transverse-transverse) third-order structure function was not calculated from the measured data.† The relation (2.19) could have been used without any extra measurements, as a sensitive test of the accuracy of the measured dissipation and at the same time of the isotropy of the third-order structure tensor function. We also note that (2.16) can hold without (1.2) and (2.19) holding separately, since (2.16) can hold without (2.20) being the case.

If $\boldsymbol{B}^{(3)}$ is isotropic, so that (1.2) and (2.19) are valid, then the flow of energy from the large scales into the inertial range will be equally distributed over different components, as our Fourier analysis of §3 shows. In this case the energy content will almost certainly also be equally distributed over different components in the inertial range and for smaller scales, since it is difficult to believe that there could be any mechanism driving the small scales of turbulence towards a non-isotropic state, once they have been created isotropically. We therefore suggest that the most sensitive and most appropriate test of the local isotropy hypothesis is to test (1.2) and (2.19).

† During the revision of this note the author has been communicating with Dr Saddoughi. Fortunately, the measured data have been saved. The two components of the form B_{ltt} are now under evaluation. The result will be reported in cooperation with the author of this note.

If these relations could be repeatedly confirmed, then the question would almost certainly be answered.

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Appendix. The derivation of Monin & Yaglom

An equation which is similar to equation (2.3) of §2 has been derived by Monin & Yaglom (1975 pp. 401–403). However, in this derivation it is argued that if one includes isotropy of scalar–velocity structure functions in the definition of local isotropy, then the pressure terms that appear in the derivation must by local isotropy be equal to zero. In this Appendix we show that this conclusion is not valid.

The derivation of Monin & Yaglom starts from the equation

$$\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial r_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial p_0}{\partial x_{0_i}} + \nu(\Delta + \Delta_0)(u_i - u_{0_i}), \tag{A 1}$$

where $\boldsymbol{v} = \delta \boldsymbol{u}$ is the difference between the velocities $\boldsymbol{u}$ and $\boldsymbol{u}_0$ at the points $\boldsymbol{x}$ and $\boldsymbol{x}_0$, with the separation vector $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_0$. This equation is then multiplied by v_j; the corresponding equation where i is changed to j is multiplied by v_i, and the two resulting equations are averaged and added to each other. Their final equation is the same as equation (2.3), except that the pressure terms are missing, and since no Reynolds decomposition of the velocity field is made, the mean flow gradient terms are also missing.

When (A 1) is multiplied by v_i the first pressure term can be written

$$-\frac{1}{\rho}\left\langle (u_j - u_{0_j})\frac{\partial p}{\partial x_i}\right\rangle . \tag{A 2}$$

Assuming homogeneity, (A 2) can also be written as

$$\frac{1}{\rho}\left[-\left\langle u_j \frac{\partial p}{\partial x_i}\right\rangle + \frac{\partial}{\partial r_i}\langle u_{0_j} p\rangle\right] . \tag{A 3}$$

Monin & Yaglom conclude that this term and all other terms containing pressure by local isotropy will vanish, since the isotropic form of scalar–velocity structure functions is zero (Monin & Yaglom 1975, p. 103). The only relevant scalar–velocity structure function in this context is of course the pressure–velocity structure function $\langle \delta p \delta \boldsymbol{u}\rangle$. Thus, the argument is if $\langle \delta p \delta \boldsymbol{u}\rangle$ is zero, then (A 3) must also be equal to zero.

It is evident that this inference can be valid only if (A 3) can be reduced to first-order derivatives of pressure–velocity structure functions, such as

$$\frac{1}{\rho}\frac{\partial}{\partial r_i}\langle \delta p \delta u_j\rangle . \tag{A 4}$$

It is also evident that such a reduction is impossible. This can for example be seen by letting r go to infinity. In this limit, (A 3) reduces to its first term, a single-point quantity which generally is not zero, while (A 4) is zero in the limit when r goes to infinity. Taking the limit $r \to \infty$ does not imply any lack of generality in this case, since it is only a practical way to see that (A 3) cannot be reduced to an expression of the form (A 4). The argument of Monin & Yaglom clearly requires that such a reduction is possible.

There is also another simple way to see that the argument of Monin & Yaglom is not valid. If the argument were valid, then the pressure terms in the final equation would include $\langle \delta p \delta \boldsymbol{u} \rangle$. Otherwise, it would be impossible to make the inference: $\langle \delta p \delta \boldsymbol{u} \rangle = \mathbf{0} \Rightarrow$ pressure terms equal to zero. For dimensional reasons (A 4) is the only possible form including $\langle \delta p \delta \boldsymbol{u} \rangle$. But no such term as (A 4) can appear in the final equation, which is clear from the fact that (A 4) is odd in $\boldsymbol{r}$, while the final equation is even in $\boldsymbol{r}$, as we pointed out in §2.

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